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Solution for 'Topics in complex analysis'

(15/10/2025)

H 6.1 (Examples of infinite products of functions)
Examine for which $z \in \mathbb{C}$ the following products converge absolutely and determine the largest open set $U \subset \mathbb{C}$ on which they converge locally normally:

a) $\displaystyle\prod_{n=1}^{\infty}(1+z^n)$ b) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{z^n}{n!}\right)$ c) $\displaystyle\prod_{n=1}^{\infty}\cos(2^{-n}z)$

Solution H 6.1:
Recall that by Lemma 3.5 the absolute convergence of a product of the form $\prod_{n=1}^{\infty}(1+a_n)$ is equivalent to the convergence of $\sum_{n=1}^{\infty}|a_n|$. We apply this result for a) - c).
a) By properties of the geometric series it holds that

$$\sum_{n=1}^{\infty}|z^n| < +\infty \iff |z| < 1.$$

Hence the infinite product in a) converges absolutely on $B_1(0)$. We argue that the series also converges locally normally on this set. Indeed, let $K \subset B_1(0)$ be compact. Then $c := \sup_{z\in K}|z| < 1$. Thus

$$\sum_{n=1}^{\infty}\sup_{z\in K}|z^n| \leq \sum_{n=1}^{\infty}c^n < +\infty.$$

By definition this yields local normal convergence on $B_1(0)$. Clearly this is the largest open set with this property, as local normal convergence implies absolute convergence.

b) By the same reasoning we have to examine the absolute and local normal convergence of

$$\sum_{n=1}^{\infty}\frac{z^n}{n!}.$$

We argue that it converges locally normally on $\mathbb{C}$, which implies absolute convergence for each $z \in \mathbb{C}$. If $K \subset \mathbb{C}$ is compact then

$$\sum_{n=1}^{\infty}\sup_{z\in K}\left|\frac{z^n}{n!}\right| = \sum_{n=1}^{\infty}\frac{(\sup_{z\in K}|z|)^n}{n!} = e^{(\sup_{z\in K}|z|)} - 1 < +\infty.$$

This proves the claim.

c) We show that the series

$$\sum_{n=1}^{\infty}(\cos(2^{-n}z) - 1)$$

converges locally normally on $\mathbb{C}$. To this end, we bound the difference $\cos(z) - 1$. Using the series expansion of $\cos(z)$ at $z = 0$ we see that

$$|\cos(z) - 1| = \left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n}}{(2n)!}\right| \leq |z|^2\sum_{k=0}^{\infty}\frac{|z|^k}{k!} = |z|^2 e^{|z|}.$$

Thus, if $K \subset \mathbb{C}$ is a compact set and $c := \sup_{z\in K}|z|$ we obtain

$$\sup_{z\in K}|\cos(2^{-n}z)-1| \le 4^{-n}c^2e^{2^{-n}c}.$$

Hence

$$\sum_{n=1}^{\infty}\sup_{z\in K}|(\cos(2^{-n}z)-1)| \le c^2\sum_{n=1}^{\infty}4^{-n}e^{2^{-n}c} < +\infty,$$

which shows that the product converges locally normally (and therefore pointwise absolutely) on $\mathbb{C}$. □

H 6.2 (Further practice on infinite products)
Show that the product

$$H(z) = z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

converges locally normally on $\mathbb{C}$.
Remark: One can show that the limit $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n)\right)$ exists. Then $\Gamma(z) := e^{-\gamma z}\dfrac{1}{H(z)}$ yields an alternative representation of the Γ function.

Solution H 6.2:
As we shall see below, it suffices to bound the function $z \mapsto (1-z)e^z - 1$ on $B_1(0)$. Using the series expansion of the exponential function we have

$$(1-z)e^z - 1 = \sum_{k=1}^{\infty}\frac{z^k}{k!} - \sum_{k=0}^{\infty}\frac{z^{k+1}}{k!} = \sum_{k=1}^{\infty}\frac{z^k(1-k)}{k!} = \sum_{k=2}^{\infty}\frac{z^k(1-k)}{k!}.$$

Hence taking the modulus yields

$$|(1-z)e^z-1| \le \sum_{k=2}^{\infty}\frac{|z^k|}{(k-2)!} = |z|^2e^{|z|}.$$

Thus for $z \in B_1(0)$ we have the (non-optimal) estimate

$$|(1-z)e^z-1| \le e|z|^2.$$

Next fix a compact set $K \subset \mathbb{C}$. Then there exists $n(K) \in \mathbb{N}$ such that for all $n \ge n(K)$ we have $-z/n \in B_1(0)$ for all $z \in K$. We conclude that

$$\sum_{n\ge n(K)}\sup_{z\in K}\left|\left(1+\frac{z}{n}\right)e^{-z/n}-1\right| \le \sup_{z\in K}|z|^2e\sum_{n\ge n(K)}^{\infty}\frac{1}{n^2} < +\infty.$$

By definition this shows the local normal convergence of the infinite product H on all of $\mathbb{C}$. □

H 6.3 (The product formula for the sine)
The goal of this exercise is to derive the product formula

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \quad \forall z \in \mathbb{C}.$$

a) Use the partial fraction decomposition $\dfrac{\pi^2}{\sin^2(\pi z)} = \displaystyle\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}$ (cf. Exercise H 4.2) to show

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} \qquad \forall z \in \mathbb{C}\setminus\mathbb{Z},$$

with a suitable notion of convergence for the series on the right hand side.
Hint: You may use without proof the following fact from analysis: if $D \subset \mathbb{C}$ is a domain and $f_n, f : D \to \mathbb{C}$ are continuously differentiable functions such that $f_n(z_0) \to f(z_0)$ for some $z_0 \in D$ and $f_n' \to f'$ locally uniformly, then $f_n \to f$ locally uniformly.

b) Define the functions $f_n : \mathbb{C} \to \mathbb{C}$ by $f_n(z) = 1 - \frac{z^2}{n^2}$ and compute the logarithmic derivative f'/f for $f(z) = \pi z \prod_{n=1}^{\infty} f_n(z)$.

c) Conclude by comparing suitable terms.

Solution H 6.3:
a) Recall that $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Thus

$$(\pi \cot(\pi z))' = \pi^2 \frac{-\sin^2(\pi z) - \cos^2(\pi z)}{\sin^2(\pi z)} = -\frac{\pi^2}{\sin^2(\pi z)}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Inserting the partial fraction decomposition of Exercise H 4.2, we deduce that for $z \in \mathbb{C} \setminus \mathbb{Z}$ it holds

$$(\pi \cot(\pi z))' = -\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} = -\frac{1}{z^2} - \sum_{n \in \mathbb{N}} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right) = -\frac{1}{z^2} - 2 \sum_{n \in \mathbb{N}} \frac{z^2 + n^2}{(z^2 - n^2)^2}.$$

As shown in Exercise H 4.2 b), the series in the right hand side converges locally uniformly (and locally normally, which allows us to rearrange terms as above) on $\mathbb{C} \setminus \mathbb{Z}$. Since $\mathbb{C} \setminus \mathbb{Z}$ is a domain, we can apply the hint to the sequence $g_k : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ given by

$$g_k(z) = \frac{1}{z} + \sum_{n=1}^{k} \frac{2z}{z^2 - n^2}.$$

However it is quite difficult to evaluate the limit of $g_k(z_0)$ for any $z_0 \in \mathbb{C} \setminus \mathbb{Z}$ as $k \to +\infty$. Hence we use the hint in an abstract way. Note that for $z \in \mathbb{C} \setminus \mathbb{Z}$ we have

$$g_k'(z) = -\frac{1}{z^2} + 2 \sum_{n=1}^{k} \frac{z^2 - n^2 - 2z^2}{(z^2 - n^2)^2} = -\frac{1}{z^2} - 2 \sum_{n=1}^{k} \frac{z^2 + n^2}{(z^2 - n^2)^2}.$$

Hence $g_k'(z) \to (\pi \cot(\pi z))'$ locally uniformly for $z \in \mathbb{C} \setminus \mathbb{Z}$. Moreover, note that the sequence $g_k(\frac{1}{2})$ is bounded and decreasing. Hence there exists $c \in \mathbb{R}$ such that $g_k(\frac{1}{2}) \to c$ as $k \to +\infty$. Applying the hint, we deduce that

$$g_k(z) \to \pi \cot(\pi z) + \underbrace{c - \pi \cot(\tfrac{\pi}{2})}_{=: c_0} \tag{1}$$

locally uniformly for $z \in \mathbb{C} \setminus \mathbb{Z}$, so that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} - c_0. \tag{2}$$

We show that $c_0 = 0$. To this end, note that $\pi \cot(\pi z)$ has a first order pole at $z = 0$. Hence its Laurent series at the origin reads

$$\pi \cot(\pi z) = \sum_{j=-1}^{\infty} a_j z^j.$$

From (2) it follows that $a_{-1} = 1$ and $a_0 = c_0$. Another formula for a_0 is given by

$$a_0 = \frac{\mathrm{d}}{\mathrm{d}z}(z\pi\cot(\pi z))|_{z=0} = \lim_{z\to 0}\pi\left(\cot(\pi z) - \frac{\pi z}{\sin^2(\pi z)}\right) = \pi\lim_{z\to 0}\left(\frac{\cos(\pi z)\sin(\pi z) - \pi z}{\sin^2(\pi z)}\right).$$

Since $\cos(\pi z) = 1+\mathcal{O}(z^2)$ and $\sin(\pi z) = \pi z+\mathcal{O}(z^3)$ it follows that $\cos(\pi z)\sin(\pi z)-\pi z = \mathcal{O}(z^3)$, so that the above equality yields $a_0 = c_0 = 0$ as claimed.

b) Note that for $z \in \mathbb{C}\setminus\{\pm n\}$ we have

$$\frac{f_n'(z)}{f_n(z)} = \frac{2z}{z^2 - n^2}.$$

We next argue that the product $f(z) = \pi z\prod_{n=1}^{\infty} f_n(z)$ converges locally normally on $\mathbb{C}$. Indeed, let $K \subset \mathbb{C}$ be compact. Then

$$\sum_{n=1}^{\infty}\sup_{z\in K}\left|\frac{z^2}{n^2}\right| \le \sup_{z\in K}|z|^2\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty.$$

Hence we can apply Proposition 3.12 so that

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{f_n'(z)}{f_n(z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}$$

for all $z \in \mathbb{C}\setminus\mathbb{Z}$, and the series converges locally normally there. We used that $Z(f) = \mathbb{Z}$, by Lemma 3.11.

c) Note that the logarithmic derivative of $h(z) = \sin(\pi z)$ on $\mathbb{C}\setminus\mathbb{Z}$ is given by $\pi\cot(\pi z)$. Hence items a) and b) imply that

$$\frac{h'}{h} = \frac{f'}{f}$$

on the domain $\mathbb{C}\setminus\mathbb{Z}$. In particular, we deduce that on this domain

$$\left(\frac{h}{f}\right)' = \frac{h'f - hf'}{f^2} = 0.$$

Thus there exists a constant $\hat{c} \in \mathbb{C}$ such that $h = \hat{c}f$ on $\mathbb{C}\setminus\mathbb{Z}$ and by the identity theorem this extends to the whole complex plane. In order to conclude we have to show that $\hat{c} = 1$. To this end, note that for $z \neq 0$ we can write

$$\frac{\sin(\pi z)}{\pi z} = \hat{c}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Letting $z \to 0$ we deduce that $\hat{c} = 1$, which finishes the proof. □

H 6.4 (Consequences of the sine product formula)
Use the product formula from Exercise H 6.3 to show the following statements:

a) $\displaystyle\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$ b) $\displaystyle\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$ c)* $\displaystyle\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$

Hint: For c) use a Taylor expansion.

Solution H 6.4:
The product formula reads

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big).$$

a) We insert $z = \frac{1}{2}$ and obtain

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{4n^2 - 1}{4n^2}.$$

Taking the inverse of the infinite product (which is allowed term by term as the limit is not zero) we deduce the claimed formula.

b) Note that by the double angle formula $\sin(2z) = 2\sin(z)\cos(z)$ (this is well-known for $z \in \mathbb{R}$ and extends to $z \in \mathbb{C}$ by the identity theorem) we have

$$\begin{aligned}\sin(\pi z)\cos(\pi z) =& \frac{1}{2} 2\pi z \prod_{n=1}^{\infty} \Big(1 - \frac{4z^2}{n^2}\Big) = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{4z^2}{(2n)^2}\Big) \prod_{n=1}^{\infty} \Big(1 - \frac{4z^2}{(2n-1)^2}\Big) \\ =& \sin(\pi z) \prod_{n=1}^{\infty} \Big(1 - \frac{4z^2}{(2n-1)^2}\Big)\end{aligned}$$

Hence the claim follows from the identity theorem.

c) Let us rewrite the product formula as a Taylor series. From the Taylor expansion of $\sin(\pi z)$ at $z = 0$, we have locally uniformly on $\mathbb{C}$ that

$$\begin{aligned}\lim_{m\to+\infty} \sum_{n=1}^{m} (-1)^{n-1} \frac{(\pi z)^{2n-1}}{(2n-1)!} =& \sin(\pi z) = \lim_{m\to+\infty} \pi z \prod_{n=1}^{m} \Big(1 - \frac{z^2}{n^2}\Big) \\ =& \pi z - \lim_{m\to+\infty} \left(\pi z \sum_{n=1}^{m} \frac{z^2}{n^2} + g_m(z) \right),\end{aligned}$$

where $g_m(z)$ is the remainder. Since $\sum_{n=1}^{m} \frac{z^2}{n^2}$ converges locally uniformly on $\mathbb{C}$ as $m \to \infty$, also $g_m(z)$ converges locally uniformly on $\mathbb{C}$ to some $g : \mathbb{C} \to \mathbb{C}$. Moreover, each g_m has a zero of order at least 5 at $z = 0$, so that $g_m^{(k)}(0) = 0$ for all $0 \le k \le 4$. From Theorem 1.5 we deduce that $g^{(k)}(0) = 0$ for all $0 \le k \le 4$, so g also has a zero of order at least 5 at $z = 0$. Hence we can write

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\pi z)^{2n-1}}{(2n-1)!} = \pi z - \pi z^3 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=5}^{\infty} a_n z^n.$$

By uniqueness of Taylor expansion, the coefficients of the series above have to coincide. Comparing the coefficient of z^3 yields the claim. □